Asymptotic behaviour of tests for a unit root against an explosive alternative*

David I. Harvey and Stephen J. Leybourne
School of Economics, University of Nottingham

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Abstract

We compare the asymptotic local power of upper-tail unit root tests against an explosive alternative based on ordinary least squares (OLS) and quasi-differenced (QD) demeaning/detrending. We find that under an asymptotically negligible initialization, the QD-based tests are near asymptotically efficient and generally offer superior power to OLS-based approaches; however, the power gains are much more modest than in the lower-tail testing context. We also find that asymptotically non-negligible initial conditions do not affect the power ranking in the same way as they do for lower-tail tests, with the QD-based tests retaining a power advantage in such cases.

Keywords: Unit root testing; Explosive autoregression; Asymptotic power; Initial condition.

JEL Classification: C22; C12.

1 Introduction

Testing the null hypothesis of a unit root against a stationary alternative has received a great deal of attention in the econometrics literature. Indeed, it is now a matter of regular practice in empirical time series research to conduct unit root tests such as the t-ratio tests of Dickey and Fuller (1979) [DF] and Elliott *et al.* (1996) [ERS]. While the DF approach accounts for the assumed deterministic component of the series via prior ordinary least squares (OLS) demeaning or detrending, ERS demonstrate that gains in power are available by instead demeaning or detrending based on a quasi-differenced (QD) transformation of the regression. Assuming a negligible initialization

*Correspondence to: David Harvey, School of Economics, University of Nottingham, University Park, Nottingham, NG7 2RD, United Kingdom. Email: dave.harvey@nottingham.ac.uk

for the stochastic process, ERS show that the QD version of the DF test is near asymptotically efficient, lying arbitrarily close to the Gaussian local power envelope. However, Müller and Elliott (2003) show that the superiority of tests based on QD demeaning or detrending compared to their OLS demeaned or detrended counterparts does not carry over to the case where the initial value of the series is asymptotically non-negligible; here, the power ranking is reversed for large initial conditions.

While tests of the unit root null have predominantly been directed towards the stationary alternative, there has been growing interest in testing against an *explosive* alternative, particularly in the analysis of financial time series where explosive autoregressive behaviour can act as a model for an economic bubble. For example, Phillips *et al.* (2011) make use of forward recursive upper-tail OLS-based DF tests to determine whether the Nasdaq stock price index displayed explosive bubble-type behaviour in the 1990s. Given the differential behaviour of OLS- and QD-demeaned/detrended tests already established when testing in the lower tail, in this paper we consider whether similar features are manifest when tests are implemented in the upper tail, i.e. whether upper-tail QD-based DF tests attain higher levels of power than OLS-based tests under an assumption of an asymptotically negligible initial condition, and whether asymptotically non-negligible initial conditions create a reversal of this power ranking. We find that under an asymptotically negligible initialization, the upper-tail QD-based tests are again near asymptotically efficient and generally offer superior power to tests based on OLS demeaning/detrending; however, the power gains are much more modest than in the lower-tail testing context. Moreover, we find that asymptotically non-negligible initial conditions do not affect the power ranking in the same way as for lower-tail tests, with the QD-based variants retaining a power advantage in this setting.

2 The model and test statistics

We consider a DGP given by

$$\begin{aligned} y_t &= \mu + \beta t + u_t, \quad t = 1, ..., T && (1) \\ u_t &= \rho_T u_{t-1} + \varepsilon_t, \quad t = 2, ..., T. && (2) \end{aligned}$$

where ε_t is a martingale difference sequence with conditional variance σ^2 and $\sup_t E(\varepsilon_t^4) < \infty$. We assume $\rho_T = 1 + c/T$, where c is a finite constant. We consider two cases for the initial condition, modelling u_1 as either asymptotically negligible via Assumption 1: $u_1 = o_p(T^{-1/2})$, or asymptotically non-negligible via Assumption 2: $u_1 = T^{1/2}\sigma\alpha$, where $\alpha \neq 0$ is a finite constant, cf. Müller and Elliott (2003).[1]

Our interest in this paper centres on discriminating between the unit root null hypothesis $H_0 : \rho_T = 1$ $(c = 0)$ and either the local-to-unit root *stationary* alternative $H_S : \rho_T < 1$ $(c < 0)$ or the local-to-unit root *explosive* alternative $H_E : \rho_T > 1$ $(c > 0)$. The unit root tests we consider are the t-ratio test of DF based on OLS demeaning or

[1] Note that (1)-(2) restricts the deterministic constant/trend to enter linearly, and ensures that any locally explosive behaviour in y_t arises from the stochastic component u_t alone, cf. Phillips, Shi and Yu (2013).

detrending (denoted by $DF\text{-}OLS^{\mu}$ and $DF\text{-}OLS^{\tau}$ respectively) and the DF-type t-ratio test of ERS based on QD demeaning or detrending (denoted by $DF\text{-}QD^{\mu}$ and $DF\text{-}QD^{\tau}$ respectively).

The $DF\text{-}OLS^{i}$ test ($i = \mu,\ \tau$) is based on the t-statistic for testing $\rho = 1$ in the fitted regression equation

$$\hat{u}_t = \rho \hat{u}_{t-1} + e_t, \quad t = 2, ..., T \tag{3}$$

where $\hat{u}_t := y_t - z_t'\hat{\theta}$ is the residual from an OLS regression of y_t on $z_t := 1$, $\theta = \mu$ ($DF\text{-}OLS^{\mu}$) or $z_t := (1, t)'$, $\theta = (\mu, \beta)'$ ($DF\text{-}OLS^{\tau}$). The corresponding $DF\text{-}QD^{i}$ test ($i = \mu,\ \tau$) is based on the t-statistic for testing $\rho = 1$ in the fitted regression

$$\tilde{u}_t = \rho \tilde{u}_{t-1} + e_t, \quad t = 2, ..., T \tag{4}$$

where, on setting $\bar{\rho}_T := 1 + \bar{c}/T$ for some chosen constant $\bar{c}$, $\tilde{u}_t := y_t - z_t'\tilde{\theta}$, where $\tilde{\theta}$ is obtained from the QD regression of $\mathbf{y}_{\bar{c}} := (y_1, y_2 - \bar{\rho}_T y_1, ..., y_T - \bar{\rho}_T y_{T-1})'$ on $\mathbf{Z}_{\bar{c}} := (z_1, z_2 - \bar{\rho}_T z_1, ..., z_T - \bar{\rho}_T z_{T-1})'$, where $z_t := 1$ for $DF\text{-}QD^{\mu}$, and $z_t := (1, t)'$ for $DF\text{-}QD^{\tau}$. When the $DF\text{-}OLS^{\mu}$ and $DF\text{-}QD^{\mu}$ tests are considered, the implicit assumption is that $\beta = 0$ in (1).

3 Asymptotic behaviour

The asymptotic properties of the four tests, assuming $\beta = 0$ in (1) for $DF\text{-}OLS^{\mu}$ and $DF\text{-}QD^{\mu}$, are as follows:

$$DF\text{-}OLS^{\mu} \xrightarrow{d} \frac{K_c^{\mu}(1)^2 - K_c^{\mu}(0)^2 - 1}{2\sqrt{\int_0^1 K_c^{\mu}(r)^2 dr}}, \qquad DF\text{-}QD^{\mu} \xrightarrow{d} \frac{K_c(1)^2 - 1}{2\sqrt{\int_0^1 K_c(r)^2 dr}},$$

$$DF\text{-}OLS^{\tau} \xrightarrow{d} \frac{K_c^{\tau}(1)^2 - K_c^{\tau}(0)^2 - 1}{2\sqrt{\int_0^1 K_c^{\tau}(r)^2 dr}}, \qquad DF\text{-}QD^{\tau} \xrightarrow{d} \frac{K_c^{\tau,\bar{c}}(1)^2 - 1}{2\sqrt{\int_0^1 K_c^{\tau,\bar{c}}(r)^2 dr}},$$

where

$$\begin{aligned} K_c^{\mu}(r) &:= K_c(r) - \int_0^1 K_c(s)ds, \\ K_c^{\tau}(r) &:= K_c^{\mu}(r) - 12\left(r - \frac{1}{2}\right)\int_0^1 \left(s - \frac{1}{2}\right) K_c(s)ds, \\ K_c^{\tau,\bar{c}}(r) &:= K_c(r) - r\left\{\bar{c}^* K_c(1) + 3(1 - \bar{c}^*)\int_0^1 sK_c(s)dr\right\} \end{aligned}$$

with $\bar{c}^* := (1 - \bar{c})/(1 - \bar{c} + \bar{c}^2/3)$ and

$$K_c(r) := \begin{cases} W_c(r) & \text{under Assumption 1} \\ \alpha(e^{rc} - 1) + W_c(r) & \text{under Assumption 2} \end{cases}$$

where $W_c(r) := \int_0^r e^{(r-s)c} dW(s)$ and $W(r)$ is a standard Wiener process. These limit distributions follow directly from results in Müller and Elliott (2003).[2]

[2]Note that these limit results continue to hold in the case of serially correlated ε_t, provided appropriate lagged difference augmentation is applied to the DF-type regressions (3) and (4).

The asymptotic distributions of the tests under H_0 are found by setting $c = 0$, at which value $K_c(r) = W(r)$ under both Assumptions 1 and 2, and so the initial condition plays no role. For the stationary alternative H_S we require lower-tail asymptotic critical values, while for the explosive alternative H_E it is upper-tail asymptotic critical values that are needed. Under Assumption 1, the initial condition is asymptotically negligible and has no impact on the limit distributions of the tests. Under Assumption 2, however, under either alternative hypothesis (i.e. when $c \neq 0$), the initial condition does have an effect in the limit.

ERS choose $\bar{c}$ such that when testing H_0 against H_S, the Gaussian point optimal invariant test of $c = 0$ against $c = \bar{c}$, which forms the asymptotic Gaussian local power envelope, has a power of 0.50. For a nominal 0.05-level test, this yields the (approximate) values $\bar{c} = -7$ and $\bar{c} = -13.5$ for $DF\text{-}QD^{\mu}$ and $DF\text{-}QD^{\tau}$, respectively. We repeated this exercise in the context of testing H_0 against H_E, and found the (approximate) values $\bar{c} = 1.6$ and $\bar{c} = 2.4$ for $DF\text{-}QD^{\mu}$ and $DF\text{-}QD^{\tau}$, respectively, and these are adopted in what follows. Asymptotic critical values for testing against H_S are already documented; for testing against H_E, Table 1 reports asymptotic critical values at conventional significance levels for the four tests we consider. Here and throughout the paper, numerical results are obtained by direct simulation of the limiting distributions, approximating the Wiener processes using $NIID(0, 1)$ random variates, and with the integrals approximated by normalized sums of 2000 steps. The simulations were programmed in Gauss 9.0 using 50,000 Monte Carlo replications.

3.1 Asymptotically negligible initial conditions

For the case of Assumption 1, where the initial condition has no asymptotic effect, Figure 1 plots the local asymptotic power for lower-tail tests of H_0 against H_S, conducted at the 0.05 level, across $c \leq 0$, together with the Gaussian local power envelope. We report results for $c = \{0, -0.5, -1.0, ..., -30.0\}$ so that the envelope powers range from 0.05 at $c = 0$ to values in excess of 0.995. We observe the familiar ERS result showing that, in Figure 1(a), the power of $DF\text{-}QD^{\mu}$ (effectively) coincides with the power envelope across all c and is substantially higher than that of $DF\text{-}OLS^{\mu}$, while in Figure 1(b), the power of $DF\text{-}QD^{\tau}$ again coincides with the power envelope and is higher than that of $DF\text{-}OLS^{\tau}$.

Figure 2 examines whether these results continue to hold for upper-tail tests of H_0 against H_E, across $c \geq 0$. Here we consider $c = \{0, 0.1, 0.2, ..., 7.0\}$ so that the envelope covers the same range of powers as in the lower-tail case. Figure 2(a) shows that $DF\text{-}QD^{\mu}$ coincides with the power envelope for values of c up to approximately 1.5, after which it falls very slightly below the power envelope. For values up to about 2.5, $DF\text{-}OLS^{\mu}$ is less powerful than $DF\text{-}QD^{\mu}$ but thereafter $DF\text{-}OLS^{\mu}$ is actually closer to the power envelope than $DF\text{-}QD^{\mu}$. While, on balance, it is clear that $DF\text{-}OLS^{\mu}$ is a less powerful test than $DF\text{-}QD^{\mu}$, the magnitudes of the loss are considerably smaller here than for the corresponding lower-tail tests of H_0 against H_S seen in Figure 1(a). In Figure 2(b), we see that the power of $DF\text{-}QD^{\tau}$ coincides with the power envelope and, while $DF\text{-}OLS^{\tau}$ is less powerful, it is only marginally less so, with the powers being much closer than for the lower-tail tests of H_0 against H_S shown in Figure 1(b).

Overall, we conclude that whether lower-tail or upper-tail tests are being considered, the *DF-QD*$^\mu$ and *DF-QD*$^\tau$ tests should be thought of as preferable to the corresponding *DF-OLS*$^\mu$ and *DF-OLS*$^\tau$ tests, as the former almost never deviate from the power envelope. However, the case for this preference is rather weaker for upper-tail testing than it is for lower-tail testing.

3.2 Asymptotically non-negligible initial conditions

We now turn to the case of Assumption 2, where the initial condition does have an asymptotic influence on the local power of the tests. Figures 3(a) and 3(b) show asymptotic local powers for lower-tail tests of H_0 against H_S, across $\alpha \geq 0$ when $c = -7$ and $c = -13.5$, respectively; these are the values of c that yield an (approximate) asymptotic power of 0.50 for *DF-QD*$^\mu$ and *DF-QD*$^\tau$ under Assumption 1. Results are reported for $\alpha = \{0, 0.02, 0.04, ..., 1.20\}$.[3] Again we observe the familiar Müller-Elliott pattern that the power profiles of *DF-QD*$^\mu$ and *DF-QD*$^\tau$ exhibit monotonic decrease in α, whilst the powers of *DF-OLS*$^\mu$ and *DF-OLS*$^\tau$ demonstrate precisely the reverse form of behaviour. The powers of *DF-QD*$^\mu$ and *DF-QD*$^\tau$ are effectively zero for $\alpha > 0.6$ and 0.7, respectively; it is this unappealing feature of the *DF-QD*$^\mu$ and *DF-QD*$^\tau$ tests that makes it highly questionable whether, when little is known about the initial condition (as is typically the case in practice), they should be considered a better option than *DF-OLS*$^\mu$ and *DF-OLS*$^\tau$.

Figures 4(a) and 4(b) show local asymptotic powers for upper-tail tests of H_0 against H_E when $c = 1.6$ and $c = 2.4$, respectively, these values of c being those that give (approximate) power of 0.50 for *DF-QD*$^\mu$ and *DF-QD*$^\tau$ under Assumption 1. The standout feature is that the powers of *DF-QD*$^\mu$ and *DF-QD*$^\tau$ are here monotonically *increasing* in α, as are their counterparts *DF-OLS*$^\mu$ and *DF-OLS*$^\tau$. Moreover, the powers of the *DF-QD*$^\mu$ and *DF-QD*$^\tau$ tests consistently exceed those of the corresponding *DF-OLS*$^\mu$ and *DF-OLS*$^\tau$ tests across α.

Clearly then, asymptotically non-negligible initial conditions do not reproduce their pernicious power effects on *DF-QD*$^\mu$ and *DF-QD*$^\tau$ from lower-tail testing of H_0 against H_S once the context changes to upper-tail testing of H_0 against H_E.[4] Therefore, for this latter testing problem the *DF-QD*$^\mu$ and *DF-QD*$^\tau$ pair of tests emerge, pretty much without contention, as the tests of choice.

4 Conclusion

The results of this paper show that when the initial condition of a time series is asymptotically negligible, upper-tail QD-based DF unit root tests are near asymptotically efficient, as has been observed previously for their lower-tail counterparts. They also

[3]Note that the limit distributions of the tests are invariant to the sign of α.

[4]Unreported simulations confirm that similar qualitative patterns of results are obtained for different values of c.

generally possess superior local asymptotic power properties to tests based on OLS demeaning or detrending, although the power gains that the QD approach delivers are considerably less significant than for lower-tail testing. Interestingly, when the initial condition is asymptotically non-negligible, the power of the upper-tail QD-based tests exceeds that of the corresponding OLS-based tests, and is increasing with the magnitude of the initial value, in complete contrast to what is found for lower-tail unit root tests. It seems clear, therefore, that when conducting unit root tests against an explosive alternative, QD demeaning or detrending does not suffer from the same drawbacks that are associated with testing against a stationary alternative, and worthwhile gains are available compared to OLS demeaning or detrending, regardless of the precise specification of the process's initialization. For these reasons, there is little need for a parallel development of lower-tail procedures that attempt to exploit differences in QD and OLS approaches across initial conditions, such as the union of rejections-based procedures of Harvey *et al.* (2009). Overall, QD demeaning or detrending offers an improved approach for practitioners testing for explosive behaviour in economic and financial time series, notwithstanding the fact that the potential gains are quite modest.

Finally, we also investigated whether such gains translate to related testing approaches such as the recursive upper-tail DF-type tests of Phillips *et al.* (2011) for detecting asset price bubbles. Specifically, we simulated the large sample power performance of OLS and QD demeaned and detrended versions of the Phillips *et al.* (2011) test, using (1)-(2) with the same settings as underlie Figures 2 and 4. As might be expected, we found the relative behaviour of the OLS and QD recursive tests to be qualitatively very similar to that observed for the non-recursive test results reported in the figures, thus the findings of this paper apply also to recursive-based procedures.[5]

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[5]Full details of these simulations are available from the authors on request.

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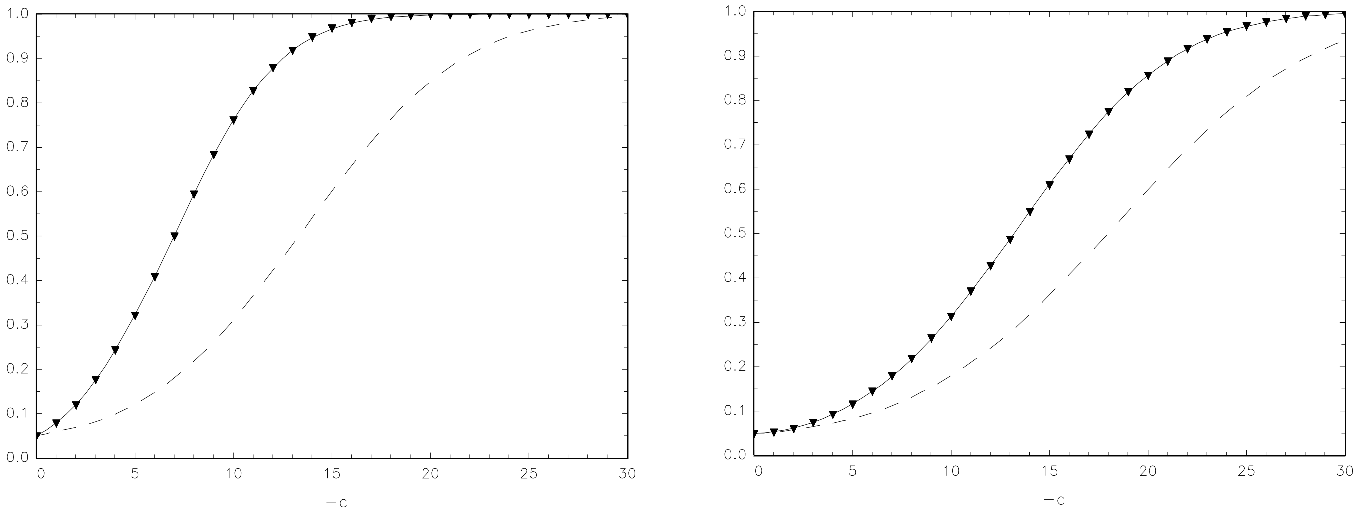
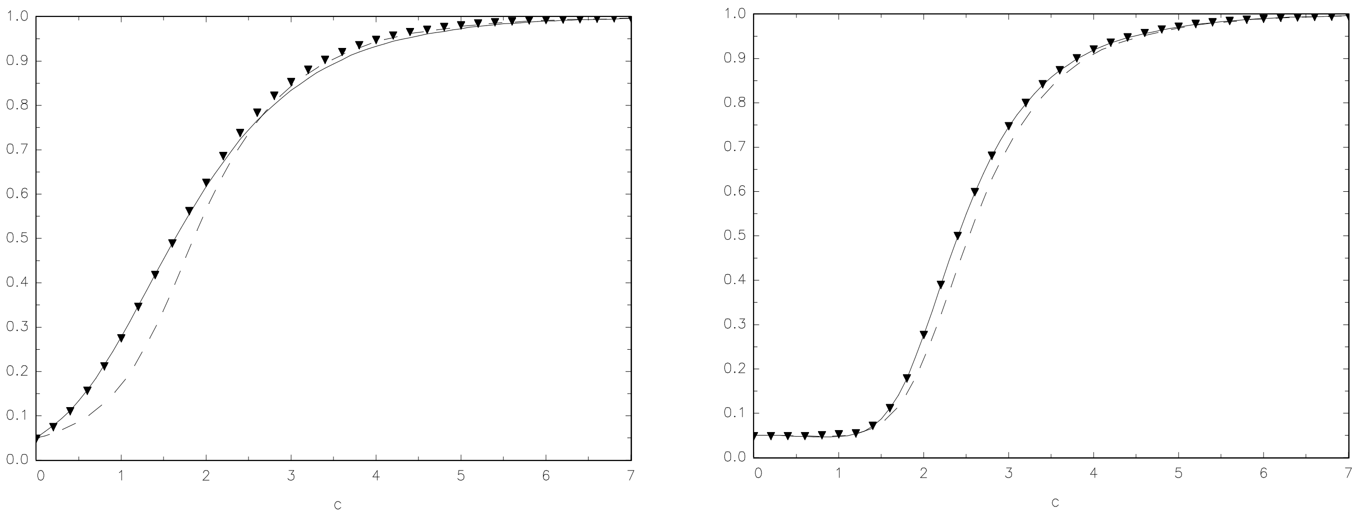
Table 1. Asymptotic ξ-level critical values for upper-tail unit root tests

	$\xi = 0.10$	$\xi = 0.05$	$\xi = 0.01$
$DF\text{-}OLS^{\mu}$	-0.44	-0.08	0.61
$DF\text{-}OLS^{\tau}$	-1.24	-0.94	-0.32
$DF\text{-}QD^{\mu}$	0.89	1.28	2.00
$DF\text{-}QD^{\tau}$	2.64	3.31	4.59



(a) $i = \mu$ (b) $i = \tau$

Figure 1. Asymptotic local power of lower-tail unit root tests, $u_1 = o_p(T^{-1/2})$:
Envelope: ▼ ▼ ▼ , *DF-OLS*^{i}: $--$, *DF-QD*^{i}: —



(a) $i = \mu$ (b) $i = \tau$

Figure 2. Asymptotic local power of upper-tail unit root tests, $u_1 = o_p(T^{-1/2})$:
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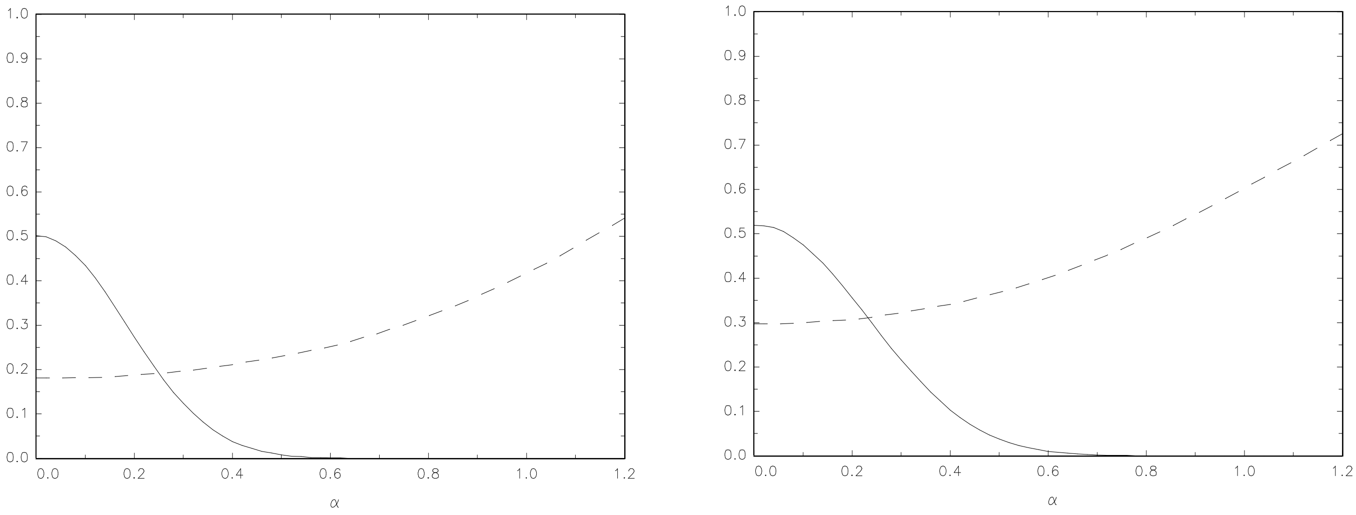
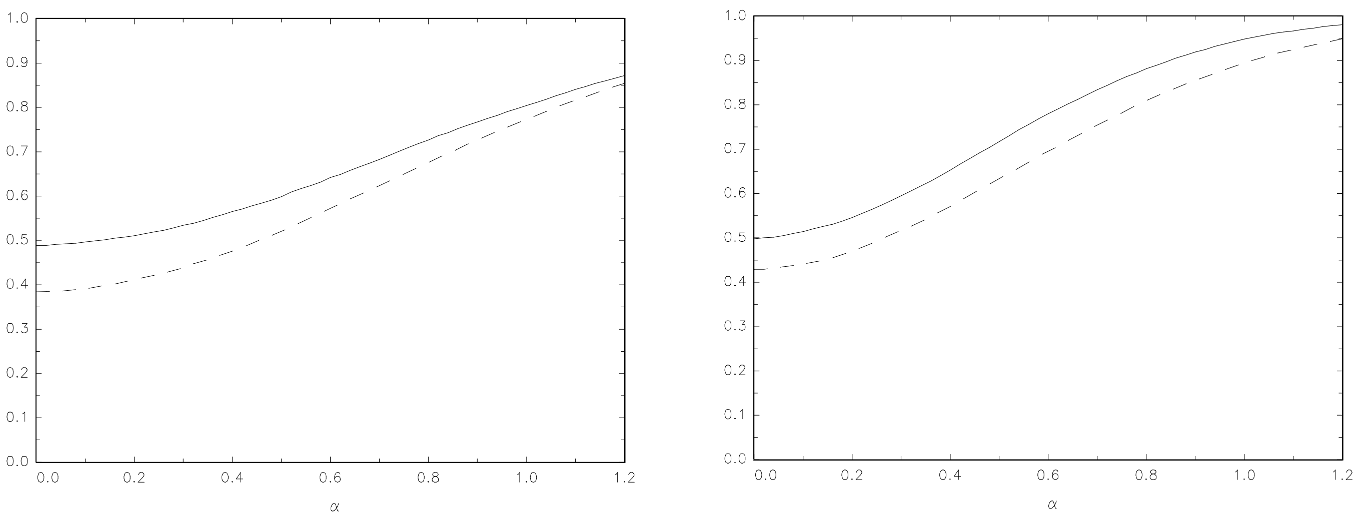


(a) $i = \mu$, $c = -7$ (b) $i = \tau$, $c = -13.5$

Figure 3. Asymptotic local power of lower-tail unit root tests, $u_1 = T^{1/2}\sigma\alpha$:
$DF\text{-}OLS^i$: $--$, $DF\text{-}QD^i$: —



(a) $i = \mu$, $c = 1.6$ (b) $i = \tau$, $c = 2.4$

Figure 4. Asymptotic local power of upper-tail unit root tests, $u_1 = T^{1/2}\sigma\alpha$:
$DF\text{-}OLS^i$: $--$, $DF\text{-}QD^i$: —